# Normal projections in Krein spaces

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**Abstract**

Given a complex Krein space $\mathcal{H}$ with fundamental symmetry $J$, the aim of this note is to characterize the set of $J$-normal projections

$$\mathcal{Q} = \{Q \in L(\mathcal{H}) : \ Q^2 = Q \text{ and } Q^{\#}Q = QQ^{\#}\}.$$

The ranges of the projections in $\mathcal{Q}$ are exactly those subspaces of $\mathcal{H}$ which are pseudo-regular. For a fixed pseudo-regular subspace $\mathcal{S}$, there are infinitely many $J$-normal projections onto it, unless $\mathcal{S}$ is regular. Therefore, most of the material herein is devoted to parametrizing the set of $J$-normal projections onto a fixed pseudo-regular subspace $\mathcal{S}$.

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## 1 Introduction

It is well-known that a (linear, bounded) projection $Q$, acting on a Hilbert space $\mathcal{H}$, is normal ($QQ^* = Q^*Q$) if and only if it is selfadjoint ($Q = Q^*$). Therefore, there is a one-to-one correspondence between the closed subspaces of $\mathcal{H}$ and the normal projections acting on $\mathcal{H}$.

On the other hand, if $\mathcal{K}$ is a Krein space with fundamental symmetry $J$, it is easy to find $J$-normal projections which are not $J$-selfadjoint (see Example 1 in Section 3). For a fixed Krein space $\mathcal{K}$ with fundamental symmetry $J$, the purpose of this work is to describe those projections acting on $\mathcal{K}$ which are $J$-normal, i.e. those $Q = Q^2 \in L(\mathcal{K})$ satisfying

$$QQ^{\#} = Q^{\#}Q,$$

where $Q^{\#}$ is the $J$-adjoint of $Q$.

If $Q$ is $J$-normal, observe that $E = QQ^{\#}$ is a $J$-selfadjoint projection whose range, hereafter denoted by $R(E)$, is contained in $R(Q)$. Thus, $R(Q)$ contains a regular subspace of $\mathcal{K}$. On the other hand, $P = Q(I - Q^{\#})$ is a projection with $R(P) = R(Q) \cap R(Q)^{[\perp]} = R(Q)^{\circ}$, i.e. $R(P)$ is the isotropic part of $R(Q)$.

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Also, since $EP = PE = 0$ it follows that $Q = E + P$ and

$$R(Q) = R(E)[\dot{+}]R(P) = R(E)[\dot{+}]R(Q)^\circ,$$

that is, $R(Q)$ is a *pseudo-regular* subspace of $\mathcal{K}$, see [9] for the terminology. Conversely, it will be shown that every pseudo-regular subspace of $\mathcal{K}$ admits a $J$-normal projection onto it. However, it is not hard to prove that a pseudo-regular subspace may admit infinitely many $J$-normal projections onto it (see Example 2 in Section 4).

The importance of pseudo-regular subspaces lies in the fact that they enable to generalize some Pontryagin spaces arguments to general Krein spaces, see [9]. They have also been used as a technical tool for the study of spectral functions (and distributions) for particular classes of operators in Krein spaces [10, 11, 13, 14] and to extend the Beurling-Lax theorem for shifts in indefinite metric spaces [4, 5].

Along this work, different characterizations of $J$-normal projections will be developed. Furthermore, for a fixed pseudo-regular subspace $\mathcal{S}$, we will present a parametrization for the set of $J$-normal projections onto $\mathcal{S}$.

In the next section we introduce the basic notations and terminology used in the paper. Section 3 is devoted to describe $J$-normal projections. In particular, it is shown that every $J$-normal projection $Q$ admits a unique decomposition $Q = E + P$ where $E$ is $J$-selfadjoint and $P$ is a $J$-normal projection with $J$-neutral range. Then, the main consequences of this decomposition are discussed.

In Section 4 it is shown that a (closed) subspace $\mathcal{S}$ is the range of a $J$-normal projection if and only if it is pseudo-regular, i.e. if $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed. Then, although there is not a unique $J$-normal projection onto an arbitrary pseudo-regular subspace $\mathcal{S}$, a formula for a particular $J$-normal projection onto $\mathcal{S}$ is presented (depending only on the fundamental symmetry $J$ and the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$).

Section 5 deals with $J$-normal projections onto $J$-neutral subspaces. It will be shown that there are infinitely many $J$-normal projections onto a prescribed $J$-neutral subspace (and their nullspaces can be arbitrarily close). Then, for a fixed $J$-neutral subspace $\mathcal{N}$, a parametrization for the set of $J$-normal projections onto $\mathcal{N}$ is presented.

Finally, the aim of Section 6 is to present an explicit description of the set of $J$-normal projections onto a pseudo-regular subspace $\mathcal{S}$. First, it is shown that this set can be decomposed in a disjoint union of decks. Then, considering the projections as block-operator matrices according to an appropriate orthogonal decomposition, each deck is parametrized.

# 2 Preliminaries

**Notation and terminology** Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. The group of linear invertible operators acting on $\mathcal{H}$ is denoted by $GL(\mathcal{H})$. Also, $L(\mathcal{H})^+$ denotes the cone of positive semidefinite operators acting on $\mathcal{H}$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$.

If $T \in L(\mathcal{H}, \mathcal{K})$ then $T^* \in L(\mathcal{K}, \mathcal{H})$ denotes the adjoint operator of $T$, $R(T)$ stands for its range and $N(T)$ for its nullspace.

Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$ , $\mathcal{S} \dot{+} \mathcal{T}$ denotes the direct sum of them. On the other hand, $\mathcal{S} \oplus \mathcal{T}$ stands for their (direct) orthogonal sum and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, there exists a (unique) bounded projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$. Hereafter, it is denoted by $P_{\mathcal{S}//\mathcal{T}}$. If $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ stand for the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, $P_{\mathcal{S}//\mathcal{T}}$ can be represented as:

$$P_{\mathcal{S}//\mathcal{T}} = P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{T}})^{-1}, \tag{2.1}$$

see [2, Lemma 3.1].

Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$, the cosine of the *Friedrichs angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S} \ominus \mathcal{T}, \|x\| = 1,\, y \in \mathcal{T} \ominus \mathcal{S}, \|y\| = 1\}.$$

It is well known that

$$c(\mathcal{S}, \mathcal{T}) < 1 \quad \Leftrightarrow \quad \mathcal{S} + \mathcal{T} \text{ is closed} \quad \Leftrightarrow \quad c(\mathcal{S}^\perp, \mathcal{T}^\perp) < 1.$$

Furthermore, if $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then $c(\mathcal{S}, \mathcal{T}) < 1$ if and only if $(I - P_{\mathcal{S}})P_{\mathcal{T}}$ has closed range.

On the other hand, the *Dixmier (or minimal) angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S}, \|x\| = 1,\, y \in \mathcal{T}, \|y\| = 1\}.$$

It is clear that $c(\mathcal{S}, \mathcal{T}) \leq c_0(\mathcal{S}, \mathcal{T})$, and if $\mathcal{S} \cap \mathcal{T} = \{0\}$ then $c(\mathcal{S}, \mathcal{T}) = c_0(\mathcal{S}, \mathcal{T})$.

*Remark* 2.1. If $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then

$$c_0(\mathcal{S}, \mathcal{T}) = \|P_{\mathcal{S}} P_{\mathcal{T}}\|.$$

Also, $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ if and only if $\|P_{\mathcal{S}^\perp} P_{\mathcal{T}^\perp}\| < 1$. See [8] for further details.

## Krein spaces

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject see [6, 12, 1].

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a *fundamental decomposition* $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is denoted by $(\mathcal{H}, \langle\ ,\ \rangle)$.

Observe that the indefinite metric and the inner product of $\mathcal{H}$ are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[x, y] = \langle Jx, y \rangle, \qquad x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $L(\mathcal{H},\mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to the associated Hilbert spaces $(\mathcal{H},\langle\ ,\ \rangle_{\mathcal{H}})$ and $(\mathcal{K},\langle\ ,\ \rangle_{\mathcal{K}})$. Given $T\in L(\mathcal{H},\mathcal{K})$, the $J$-adjoint operator of $T$ is defined by $T^{\#}=J_{\mathcal{H}}T^*J_{\mathcal{K}}$, where $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T\in L(\mathcal{H})$ is $J$-selfadjoint if $T=T^{\#}$.

A vector $x\in\mathcal{H}$ is *$J$-positive* if $[\,x,x\,]>0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *$J$-positive* if every $x\in\mathcal{S}$, $x\neq 0$, is a $J$-positive vector. $J$-nonnegative, $J$-neutral, $J$-negative and $J$-nonpositive vectors and subspaces are defined analogously.

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *$J$-orthogonal complement* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]}=\{x\in\mathcal{H}:[\,x,s\,]=0,\ \text{for every } s\in\mathcal{S}\}.$$

Usually, $\mathcal{S}^{\circ}:=\mathcal{S}\cap\mathcal{S}^{[\perp]}$ (the *isotropic part of* $\mathcal{S}$) is a non-trivial subspace. Then, a subspace $\mathcal{S}$ of $\mathcal{H}$ is *$J$-non-degenerated* if $\mathcal{S}\cap\mathcal{S}^{[\perp]}=\{0\}$. Otherwise, it is a $J$-degenerated subspace of $\mathcal{H}$.

**Definition.** A subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is a *regular subspace* if it is the range of a $J$-selfadjoint projection, i.e. if there exists $E\in L(\mathcal{H})$ such that $E=E^2=E^{\#}$ and $R(E)=\mathcal{S}$.

Given a regular subspace $\mathcal{S}$, observe that $\mathcal{S}^{[\perp]}$ is the nullspace of the $J$-selfadjoint projection $E$ onto $\mathcal{S}$. Furthermore, if $P$ is the orthogonal projection onto $\mathcal{S}$, the orthogonal projection onto $\mathcal{S}^{[\perp]}$ coincides with $J(I-P)J$. Thus, by (2.1), it follows that

$$E=P(P+I-JPJ)^{-1}, \tag{2.2}$$

see [3] for another formula for $E$.

**Proposition 2.2** ([3])**.** *A closed subspace $\mathcal{S}$ is regular if and only if*

$$\|PJ(I-P)\|<1,$$

*or equivalently $(I-P)JPJ(I-P)\leq(1-\varepsilon)I$ for some $\varepsilon>0$, where $P$ is the orthogonal projection onto $\mathcal{S}$.*

The following result seems to be well known, however its proof is included for the sake of completeness.

**Lemma 2.3.** *Let $Q\in L(\mathcal{H})$ be a projection acting on a Krein space $\mathcal{H}$ with fundamental symmetry $J$. Then, the following conditions are equivalent:*

1. *$Q^{\#}Q=0$;*
2. *$R(Q)$ is a $J$-neutral subspace;*
3. *$PJP=0$, where $P$ is the orthogonal projection onto $R(Q)$;*

*4. the orthogonal projection $P$ onto $R(Q)$ admits the representation (according to the fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$)*

$$P = \frac{1}{2} \begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix},$$

*where $V \in L(\mathcal{H}_+, \mathcal{H}_-)$ is a partial isometry.*

*Proof.* The equivalences 1. $\leftrightarrow$ 2. $\leftrightarrow$ 3. $\leftrightarrow$ 4. and the implication 5. $\rightarrow$ 1. are easy to check. On the other hand, if $\mathcal{S} = R(Q)$ is a $J$-neutral subspace of $\mathcal{H}$ then its angular operator $V \in L(\mathcal{H}_+, \mathcal{H}_-)$ is a partial isometry. Therefore

$$\begin{aligned} \mathcal{S} &= \{(x_+, Vx_+) \in \mathcal{H}_+ \oplus \mathcal{H}_- : \ x_+ \in P_+(\mathcal{S}) = N(V)^\perp\} \\ &= \{(V^*Vu, Vu) \in \mathcal{H}_+ \oplus \mathcal{H}_- : \ u \in \mathcal{H}_+\} = R\left(\begin{bmatrix} VV^* \\ V \end{bmatrix}\right), \end{aligned}$$

see [12, Ch. 1, §8]. Then, since $V$ is a partial isometry, the operator

$$P = \frac{1}{2} \begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix},$$

satisfies $P^2 = P = P^*$, i.e. $P$ is the orthogonal projection onto $\mathcal{S}$. $\square$

# 3 Decompositions of a $J$-normal projection

Every normal projection acting on a Hilbert space is selfadjoint. However, the following example shows that there are $J$-normal projections acting on a Krein space (i.e. projections that commute with its $J$-adjoint) which are not $J$-selfadjoint.

*Example* 1. If $\mathbb{C}^3$ is endowed with the indefinite inner product $[\,x, y\,] = x_1\overline{y_1} + x_2\overline{y_2} - x_3\overline{y_3}$, where $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathbb{C}^3$, consider the projection $Q$ whose matrix representation in the canonical basis is given by

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Then, it is easy to see that

$$Q^\# = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \neq Q \quad \text{and} \quad QQ^\# = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Q^\# Q.$$

In what follows, the basic properties of $J$-normal projections are developed.

**Theorem 3.1.** *Given a projection $Q \in L(\mathcal{H})$, $Q$ is $J$-normal if and only if there exist a $J$-selfadjoint projection $E \in L(\mathcal{H})$ and a projection $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\# P = 0$ such that*

$$Q = E + P. \tag{3.1}$$

*The projections $E$ and $P$ are uniquely determined by $Q$.*

*Proof.* If $Q \in L(\mathcal{H})$ is a $J$-normal projection, then $E = QQ^\#$ is a $J$-selfadjoint projection. Notice that $P := Q(I - Q^\#)$ is also a projection and, since $I - Q$ is also $J$-normal, it holds that

$$PP^\# = Q(I - Q^\#)(I - Q)Q^\# = Q(I - Q)(I - Q^\#)Q^\# = 0.$$

In the same way, $P^\# P = 0$.

Conversely, suppose that $Q = E + P$ where $E$ is $J$-selfadjoint and $P$ is a projection satisfying $PP^\# = P^\# P = 0$. Since $Q^2 = Q$, it follows that $EP + PE = 0$. Notice that $R(E) \cap R(P) = \{0\}$. In fact, if $x \in R(E) \cap R(P)$ it is easy to see that $0 = (EP + PE)x = 2x$. So, $x = 0$. Therefore, $EP = PE = 0$ (and $EP^\# = P^\# E = 0$).

Thus, recalling that $PP^\# = P^\# P = 0$ it follows easily that $QQ^\# = Q^\# Q = E$, i.e. $Q$ is $J$-normal. Notice that $P = Q - E = Q(I - Q^\#)$. The uniqueness of this decomposition follows from the last part of the proof. □

If $Q \in L(\mathcal{H})$ is a $J$-normal projection, notice that the (uniquely) determined projections in the decomposition of Theorem 3.1 are

$$E = QQ^\# \quad \text{and} \quad P = Q(I - Q^\#). \tag{3.2}$$

Throughout this paper, $E$ and $P$ will be refered as the *regular part* and the *neutral part* of $Q$, respectively.

**Corollary 3.2.** *Let $Q \in L(\mathcal{H})$ be a $J$-normal projection. Then, $Q$ is $J$-selfadjoint if and only if $R(Q)^\circ$ is trivial.*

*Proof.* Observe that $Q$ is $J$-selfadjoint if and only if $Q = QQ^\#$, or equivalently, $P = Q(I - Q^\#) = 0$. But $R(P) = R(Q) \cap N(Q^\#) = R(Q)^\circ$. So, $P = 0$ if and only if $R(Q)^\circ = \{0\}$. □

**Corollary 3.3.** *Given a projection $Q \in L(\mathcal{H})$, $Q$ is $J$-normal if and only if*

$$Q = GH,$$

*where $G \in L(\mathcal{H})$ is a $J$-selfadjoint projection and $H \in L(\mathcal{H})$ is a $J$-normal projection with $J$-neutral kernel contained in $R(G)$. Furthermore, this factorization is unique and the projections $G$ and $H$ commute.*

*Proof.* If $Q$ is $J$-normal, then $G = I - (I - Q)(I - Q)^\#$ and $H = I - (I - Q)Q^\#$ satisfy the desired properties.

Conversely, if $Q = GH$ for a pair of projections $G$ and $H$ satisfying the assumptions, notice that $(I - G)(I - H) = 0$, or equivalently, $I + GH = G + H$. Thus,

$$I - Q = I - GH = (I - G) + (I - H),$$

$I - G$ is $J$-selfadjoint and $I - H$ satisfies $(I - H)(I - H)^\# = (I - H)^\#(I - H) = 0$. Then, by Theorem 3.1, $Q$ is $J$-normal.

The uniqueness of the factorization and the commutativity of $G$ and $H$ also follow from the above theorem. □

**Corollary 3.4.** *If $Q \in L(\mathcal{H})$ is a $J$-normal projection and $Q = E + P$ is the decomposition given by Theorem 3.1, then there exists a unique $J$-selfadjoint projection $F \in L(\mathcal{H})$ such that*

$$I - Q = F + P^{\#}. \tag{3.3}$$

*Moreover, $EF = 0$.*

*Proof.* Applying Theorem 3.1 to $I - Q$ it follows that its $J$-selfadjoint part is $F = (I-Q)(I-Q)^{\#}$ and

$$(I-Q) - F = (I-Q) - (I-Q)(I-Q)^{\#} = (I-Q)Q^{\#} = P^{\#}.$$

Furthermore, $E = QQ^{\#} = Q^{\#}Q$ and then it is obvious that $EF = 0$. □

**Lemma 3.5.** *Let $Q \in L(\mathcal{H})$ be a $J$-normal projection and consider the neutral part $P \in L(\mathcal{H})$ of $Q$. Then,*

$$R(P) = R(Q)^{\circ} \quad \textit{and} \quad R(P^{\#}) = N(Q)^{\circ}. \tag{3.4}$$

*Therefore, $R(Q)^{\circ}$ and $N(Q)^{\circ}$ have the same dimension and codimension.*

*Proof.* Indeed, if $Q$ is $J$-normal then $P = Q(I - Q^{\#}) = (I - Q^{\#})Q$ and

$$R(P) = R(Q) \cap N(Q^{\#}) = R(Q) \cap R(Q)^{[\perp]} = R(Q)^{\circ}.$$

The assertion on $R(P^{\#})$ follows analogously. Finally, notice that

$$\begin{aligned} \dim R(Q)^{\circ} &= \dim R(P) = \dim N(P)^{\perp} = \dim R(P^*) = \dim R(P^{\#}) \\ &= \dim N(Q)^{\circ}, \end{aligned}$$

and $\operatorname{codim} R(Q)^{\circ} = \dim N(P) = \dim R(P)^{\perp} = \dim N(P^*) = \dim N(P^{\#}) = \operatorname{codim} N(Q)^{\circ}$. □

*Remark* 3.6. Let $Q \in L(\mathcal{H})$ be a $J$-normal projection with decompositions $Q = E + P$ and $I - Q = F + P^{\#}$. From the $J$-normality of $Q$ and the formulas

$$E = QQ^{\#}, \quad P = Q(I - Q^{\#}), \quad F = (I-Q)(I-Q)^{\#} \quad \text{and} \quad PE = PF = 0,$$

the following facts are easily deduced:

1. $R(E) = R(Q) \cap R(Q^{\#})$ and $R(F) = N(Q) \cap N(Q^{\#})$. Moreover,

$$R(Q) = R(E) \; [\dot{+}] \; R(P) \quad \text{and} \quad N(Q) = R(F) \; [\dot{+}] \; R(P^{\#}).$$

2. Also, since $PP^{\#} = P^{\#}P = 0$, observe that $P + P^{\#}$ is a $J$-selfadjoint projection with range $R(Q)^{\circ} \dotplus N(Q)^{\circ}$. Therefore, $R(Q)^{\circ} \dotplus N(Q)^{\circ}$ is regular.

3. Finally, by the items above, notice that

$$\mathcal{H} = R(Q) \dotplus N(Q) = (R(E)[\dotplus]\, R(P)) \ \dotplus \ (R(F)[\dotplus]\, R(P^{\#})).$$

Then, if $Q$ is $J$-normal, $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = R(Q) \cap R(Q^{\#}) \ [\dotplus] \ (R(Q)^{\circ} \dotplus N(Q)^{\circ}) \ [\dotplus] \ N(Q) \cap N(Q^{\#}). \tag{3.5}$$

In fact, (3.5) is equivalent to the $J$-normality of $Q$.

**Proposition 3.7.** *Let $Q \in L(\mathcal{H})$ be a projection. Then, $Q$ is $J$-normal if and only if*

$$\mathcal{H} = R(Q) \cap R(Q^{\#}) \dotplus R(Q) \cap N(Q^{\#}) \dotplus N(Q) \cap R(Q^{\#}) \dotplus N(Q) \cap N(Q^{\#}). \tag{3.6}$$

*Proof.* If $Q$ is $J$-normal, the decomposition follows from item 3. in the above remark. Conversely, suppose that (3.6) holds. Given $x \in \mathcal{H}$ there exist (unique) $x_1 \in R(Q) \cap R(Q^{\#})$, $x_2 \in R(Q) \cap N(Q^{\#})$, $x_3 \in N(Q) \cap R(Q^{\#})$ and $x_4 \in N(Q) \cap N(Q^{\#})$ such that $x = x_1 + x_2 + x_3 + x_4$. Then,

$$Q^{\#}Qx = Q^{\#}(x_1 + x_2) = x_1 = Q(x_1 + x_3) = QQ^{\#}x.$$

Therefore, $Q^{\#}Qx = QQ^{\#}x$ for every $x \in \mathcal{H}$, i.e. $Q$ is $J$-normal. $\square$

# 4 The range of a $J$-normal projection

The aim of this section is to characterize the ranges of the family of $J$-normal projections acting on a Krein space. The main result in this direction addresses the fact that a (closed) subspace is the range of a $J$-normal projection if and only if it is a pseudo-regular subspace. Thus, the first paragraphs are devoted to recall the definition of pseudo-regularity and to state some well known equivalent conditions. Throughout this section, $\mathcal{H}$ denotes a Krein space with fundamental symmetry $J$.

**Definition.** A closed subspace $\mathcal{S}$ of $\mathcal{H}$ is called *pseudo-regular* if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed.

The following proposition compiles several conditions which are equivalent to pseudo-regularity. These facts are well known but they are scattered throughout the literature and different research papers, e.g. see [12, 5, 9, 13].

**Proposition 4.1.** *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and consider its Gramian operator $G_{\mathcal{S}} = P_{\mathcal{S}}J|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$. Then, the following conditions are equivalent:*

1. *$\mathcal{S}$ is pseudo-regular.*
2. *$(\mathcal{S}^{\circ})^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$.*
3. *There exists a regular subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{S}^{\circ}[\dotplus]\,\mathcal{M}$.*

4. *If $\mathcal{S} = \mathcal{T} \dot{+} \mathcal{S}^\circ$, then $\mathcal{T}$ is regular.*

5. *There exists a regular subspace $\mathcal{N} \supseteq \mathcal{S}$ such that $\mathcal{S}^\circ = \mathcal{N} \cap \mathcal{S}^{[\perp]}$.*

6. *$\mathcal{S}/\mathcal{S}^\circ$ is a Krein space.*

7. *$0$ is an isolated point of $\sigma(G_{\mathcal{S}})$.*

**Proposition 4.2** (T. Ando)**.** *Given a (closed) subspace $\mathcal{S}$ of $\mathcal{H}$, consider its isotropic part $\mathcal{S}^\circ$. Let $P$ and $P_0$ denote the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$, respectively. Then, $\mathcal{S}$ is pseudo-regular if and only if*

$$\|(P - P_0)J(I - P)\| < 1.$$

*Proof.* Observe that $J(I-P)J$ is the orthogonal projection onto $\mathcal{S}^{[\perp]}$. By definition, $\mathcal{S}$ is pseudo-regular if

$$\mathcal{S} + \mathcal{S}^{[\perp]} \quad \text{is closed.}$$

But $\mathcal{S}+\mathcal{S}^{[\perp]}$ is closed if and only if $c(\mathcal{S}, \mathcal{S}^{[\perp]}) < 1$. Also, notice that $c(\mathcal{S}, \mathcal{S}^{[\perp]}) = c_0(\mathcal{S} \ominus \mathcal{S}^\circ, \mathcal{S}^{[\perp]}) = \|(P - P_0)J(I - P)J\|$ (see the Preliminaries). Hence, $\mathcal{S}$ is pseudo-regular if and only if

$$\|(P - P_0)J(I - P)\| < 1. \qquad \square$$

**Theorem 4.3.** *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, $\mathcal{S}$ is the range of a $J$-normal projection if and only if $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$.*

*Proof.* If $\mathcal{S}$ is the range of a $J$-normal projection $Q$ then, by Remark 3.6, $\mathcal{S} = R(E)[\dot{+}]\mathcal{S}^\circ$ where $E = QQ^\#$. Furthermore, $R(E)$ is regular because $E$ is a $J$-selfadjoint projection. Thus, $\mathcal{S}$ is a pseudo-regular subspace.

Conversely, suppose that $\mathcal{S}$ is a pseudo-regular subspace and let $P$ be the orthogonal projection onto the isotropic subspace $\mathcal{S}^\circ$. Since $R(P)$ is $J$-neutral, it follows by Lemma 2.3 that $PJP = 0$. Then, $PP^\# = P^\# P = 0$.

Consider the subspace $\mathcal{T} = \mathcal{S} \ominus \mathcal{S}^\circ$. Since $\mathcal{S} = \mathcal{T}[\dot{+}]\mathcal{S}^\circ$, Proposition 4.1 assures that $\mathcal{T}$ is a regular subspace of $\mathcal{H}$. Thus, there is a (unique) $J$-selfadjoint projection $E$ with $R(E) = \mathcal{T}$.

Furthermore, $PE = EP = 0$ because $\mathcal{T} \subset (\mathcal{S}^\circ)^\perp$ and $\mathcal{S}^\circ \subset \mathcal{S}^{[\perp]} \subset \mathcal{T}^{[\perp]}$. Then $Q = E + P$ is also a projection with

$$R(Q) = R(E) + R(P) = \mathcal{T} \dot{+} \mathcal{S}^\circ = \mathcal{S}.$$

Finally, the $J$-normality of $Q$ follows from Theorem 3.1. $\square$

Recall that if $\kappa = \min\{\dim \mathcal{H}_+, \dim \mathcal{H}_-\} < \infty$, the Krein space with fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$ is called a *Pontryagin space* and is denoted by $\Pi_\kappa$. In a Pontryagin space $\Pi_\kappa$, a closed subspace $\mathcal{S}$ is regular if and only if it is $J$-non-degenerated (see e.g. [12]). Thus, every $J$-non-degenerated subspace of $\Pi_\kappa$ admits a (unique) $J$-selfadjoint projection onto it. Furthermore,

**Corollary 4.4.** *If $\Pi_\kappa$ is a Pontryagin space, then every closed subspace $\mathcal{S}$ of $\Pi_\kappa$ admits a $J$-normal projection onto it.*

*Proof.* Since $\mathcal{S}^\circ$ is a closed subspace of $\mathcal{S}$, $\mathcal{S}$ can be written as

$$\mathcal{S} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ).$$

Furthermore, $\mathcal{T} := \mathcal{S} \ominus \mathcal{S}^\circ$ is $J$-orthogonal to $\mathcal{S}^\circ$. Hence, $\mathcal{S} = \mathcal{S}^\circ [\dot{+}] \mathcal{T}$. It is easy to see that $\mathcal{T}$ is a $J$-non-degenerated subspace of $\mathcal{H}$ and therefore, $\mathcal{T}$ is regular because $\Pi_\kappa$ is a Pontryagin space. Thus, $\mathcal{S}$ is the direct sum of its isotropic part and a regular subspace and, by Theorem 4.3, $\mathcal{S}$ is the range of a $J$-normal projection. $\square$

The last paragraphs of this section are devoted to discussing the non-uniqueness of $J$-normal projections associated to a pseudo-regular subspace. First of all, observe the following example.

*Example* 2. As in Example 1, consider the Minkowski space $(\mathbb{C}^3, [\ ,\ ])$. Fix $\mathcal{S}$ by $\mathcal{S} = \operatorname{span}\{(1,0,0),(0,1,1)\}$. Given a vector $v = (x,y,z) \in \mathbb{C}^3 \setminus \mathcal{S}$, let $Q_v$ be the projection onto $\mathcal{S}$ along the subspace spanned by $v$. According to the canonical basis of $\mathbb{C}^3$, its matrix representation is

$$Q_v = \frac{1}{z-y} \begin{pmatrix} z-y & x & -x \\ 0 & z & -y \\ 0 & z & -y \end{pmatrix}.$$

A few calculations show that

$$Q_v^{\#} = \frac{1}{\overline{z-y}} \begin{pmatrix} \overline{z-y} & 0 & 0 \\ \overline{x} & \overline{z} & -\overline{z} \\ \overline{x} & \overline{y} & -\overline{y} \end{pmatrix}.$$

Then, it is easy to see that

$$\begin{aligned} Q_v^{\#} Q_v &= \frac{1}{|z-y|^2} \begin{pmatrix} |z-y|^2 & x\overline{(z-y)} & -x\overline{(z-y)} \\ \overline{x}(z-y) & |x|^2 & -|x|^2 \\ \overline{x}(z-y) & |x|^2 & -|x|^2 \end{pmatrix} \quad \text{and} \\ Q_v Q_v^{\#} &= \frac{1}{|z-y|^2} \begin{pmatrix} |z-y|^2 & x\overline{(z-y)} & -x\overline{(z-y)} \\ \overline{x}(z-y) & |z|^2-|y|^2 & -|z|^2+|y|^2 \\ \overline{x}(z-y) & |z|^2-|y|^2 & -|z|^2+|y|^2 \end{pmatrix}. \end{aligned}$$

Therefore, $Q_v$ is a $J$-normal projection onto $\mathcal{S}$ if and only if $|z|^2 = |x|^2 + |y|^2$.

The above example also shows that, for a fixed projection $Q \in L(\mathcal{H})$, the idempotency of the $J$-selfadjoint operators $QQ^{\#}$ and $Q^{\#}Q$ is not a sufficient condition for the $J$-normality of $Q$. In fact, notice that $Q_v^{\#}Q_v$ and $Q_vQ_v^{\#}$ are projections for every $v \in \mathbb{C}^3 \setminus \mathcal{S}$, even if $|z|^2 \neq |x|^2 + |y|^2$.

Although there is not a unique $J$-normal projection onto a fixed arbitrary pseudo-regular subspace $\mathcal{S}$, it is possible to present a particular $J$-normal projection onto $\mathcal{S}$ in terms of the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$. Observe that this particular $J$-normal projection onto $\mathcal{S}$ is the one discussed in Theorem 4.3.

**Corollary 4.5.** *Given a (closed) pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, let $P$ and $P_0$ denote the orthogonal projections onto $\mathcal{S}$ and $\mathcal{S}^\circ$, respectively. Then,*

$$Q = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1} + P_0, \tag{4.1}$$

*is a $J$-normal projection onto $\mathcal{S}$.*

*Proof.* Since $\mathcal{S} \ominus \mathcal{S}^\circ$ is a regular subspace of $\mathcal{H}$, the $J$-selfadjoint projection $E$ onto $\mathcal{S} \ominus \mathcal{S}^\circ$ can be written as

$$E = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1},$$

see (2.2). Furthermore, by Theorem 3.1, $Q = E + P_0 = (P - P_0)(P - P_0 + I - J(P - P_0)J)^{-1} + P_0$ is a $J$-normal projection onto $\mathcal{S}$. □

# 5 $J$-normal projections with $J$-neutral range

From now on, every subspace considered is assumed to be closed.

As it was shown in the previous section, a pseudo-regular subspace may admit infinitely many $J$-normal projections onto it. In order to provide a parametrization of the set of $J$-normal projections onto a prescribed pseudo-regular subspace, consider the simplest case first, i.e. a $J$-neutral subspace. This section is devoted to studying $J$-normal projections onto $J$-neutral subspaces, i.e. those projections $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\# P = 0$.

It is obvious that every $J$-neutral subspace $\mathcal{N}$ of a Krein space $\mathcal{H}$ is a pseudo-regular one, since $\mathcal{N} = \mathcal{N}^\circ$. In particular,

**Lemma 5.1.** *If $\mathcal{N}$ is a $J$-neutral subspace then the orthogonal projection $P := P_{\mathcal{N}} \in L(\mathcal{H})$ is $J$-normal. Furthermore, $PP^\# = P^\# P = 0$.*

*Proof.* By Lemma 2.3, the assumption on $\mathcal{N}$ is equivalent to $PJP = 0$. Thus,

$$PP^\# = PJP \cdot J = 0 \quad \text{and} \quad P^\# P = J \cdot PJP = 0. \qquad \square$$

**Proposition 5.2.** *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be (closed) $J$-neutral subspaces of $\mathcal{H}$ such that $\mathcal{N}_1 \cap \mathcal{N}_2 = \{0\}$. Then, the following conditions are equivalent:*

1. *there exists a $J$-normal projection $P \in L(\mathcal{H})$ such that $R(P) = \mathcal{N}_1$ and $R(P^\#) = \mathcal{N}_2$;*
2. *$\mathcal{N}_1 + \mathcal{N}_2$ is regular;*
3. *$\mathcal{N}_1 \dot{+} \mathcal{N}_2^{[\perp]} = \mathcal{H}$.*

*Proof.* *1.* $\Rightarrow$ *2.* follows from item 2. of Remark 3.6.
*2.* $\Rightarrow$ *3.*: Suppose that $\mathcal{M} = \mathcal{N}_1 + \mathcal{N}_2$ is regular. Then, $\mathcal{M}^{[\perp]} = \mathcal{N}_1^{[\perp]} \cap \mathcal{N}_2^{[\perp]}$ is also regular and

$$\mathcal{H} = \mathcal{M} \dot{+} \mathcal{M}^{[\perp]} = \mathcal{N}_1 \dot{+} (\mathcal{N}_2 \dot{+} \mathcal{N}_1^{[\perp]} \cap \mathcal{N}_2^{[\perp]}) \subseteq \mathcal{N}_1 + \mathcal{N}_2^{[\perp]},$$

because $\mathcal{N}_2$ is $J$-neutral. Analogously, $\mathcal{H} = \mathcal{N}_1^{[\perp]} + \mathcal{N}_2$ and $\mathcal{N}_1 \cap \mathcal{N}_2^{[\perp]} = (\mathcal{N}_1^{[\perp]} + \mathcal{N}_2)^{[\perp]} = \{0\}$. Thus, $\mathcal{H} = \mathcal{N}_1 \dotplus \mathcal{N}_2^{[\perp]}$.
*3.* $\Rightarrow$ *1.*: If $\mathcal{N}_1 \dotplus \mathcal{N}_2^{[\perp]} = \mathcal{H}$, consider the projection $P := P_{\mathcal{N}_1//\mathcal{N}_2^{[\perp]}}$. Then, $P^\# = P_{\mathcal{N}_2//\mathcal{N}_1^{[\perp]}}$ and it is easy to see that $PP^\# = P^\# P = 0$. Therefore, $P$ is a $J$-normal projection with $R(P) = \mathcal{N}_1$ and $R(P^\#) = \mathcal{N}_2$. $\square$

As a consequence of the above proposition, if $P$ is a $J$-normal projection onto a $J$-neutral subspace, the subspaces $R(P)$ and $R(P^\#)$ are *skewly linked* (see [12, Def. 1.29]). Moreover, in a Pontryagin space $\Pi_\kappa$, a pair of $J$-neutral subspaces $\mathcal{N}_1$, $\mathcal{N}_2$ of $\Pi_\kappa$ is skewly linked if and only if there exists a $J$-normal projection $P \in L(\mathcal{H})$ such that $R(P) = \mathcal{N}_1$ and $R(P^\#) = \mathcal{N}_2$.

*Remark* 5.3. If $\mathcal{N}$ is a $J$-neutral subspace then $\mathcal{N} + J(\mathcal{N})$ is regular. In fact, by Lemma 5.1, the orthogonal projection $P$ onto $\mathcal{N}$ is a $J$-normal projection and $R(P^\#) = J(\mathcal{N})$. So, by the above proposition, $\mathcal{N} + J(\mathcal{N})$ is regular.

**Proposition 5.4.** *Let $Q \in L(\mathcal{H})$ be a projection such that $R(Q)^\circ + N(Q)^\circ$ is regular. Then, there exist projections $E, P \in L(\mathcal{H})$ such that $PP^\# = P^\# P = 0$ and*

$$Q = E + P.$$

*Proof.* By Proposition 5.2, $\mathcal{H}$ can be decomposed as $\mathcal{H} = R(Q)^\circ + (N(Q)^\circ)^{[\perp]}$ and $P = P_{R(Q)^\circ//(N(Q)^\circ)^{[\perp]}}$ is $J$-normal. Since $R(P) \subseteq R(Q)$, it follows that $QP = P$. Also, $PQ$ is a projection and $R(PQ) = R(P)$. Furthermore,

$$\begin{aligned} N(PQ) &= N(Q) + R(Q) \cap N(P) = N(Q) + R(Q) \cap (N(Q)^\circ)^{[\perp]} \\ &\subseteq (N(Q)^\circ)^{[\perp]} = N(P). \end{aligned}$$

Thus, $PQ = P$ and $E := Q - P$ is a projection because of

$$E^2 = Q - QP - PQ + P = Q - P - P + P = Q - P = E.$$

Notice that $PE = EP = 0$ and therefore $Q = E + P$. $\square$

Following the notation of the above proof, observe that $E = Q - P = Q(I - P) = (I - P)Q$. Hence, $R(E) = R(Q) \cap N(P) = R(Q) \cap (N(Q)^\circ)^{[\perp]}$ and $N(E) = R(P) + N(Q) = R(Q)^\circ + N(Q)$. Therefore,

$$E = P_{R(Q)\cap(N(Q)^\circ)^{[\perp]}//R(Q)^\circ + N(Q)}.$$

Thus, the following is a sufficient condition to guarantee that the decomposition of the above proposition is the same as in Theorem 3.1.

**Corollary 5.5.** *Let $Q \in L(\mathcal{H})$ be a projection such that $R(Q)^\circ + N(Q)^\circ$ is regular. Then, the following conditions are equivalent:*

1. *$Q$ is $J$-normal;*
2. *$R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$;*

*3.* $N(Q) \cap (R(Q)^\circ)^{[\perp]} \subseteq N(Q) \cap N(Q^\#)$.

*Proof.* If $Q$ is $J$-normal, then $N(Q)$ is a pseudo-regular subspace. So,

$$(N(Q)^\circ)^{[\perp]} = N(Q) + N(Q)^{[\perp]} = N(Q) + R(Q^\#).$$

Then, if $x \in R(Q) \cap (N(Q)^\circ)^{[\perp]}$, there exist $u \in N(Q)$ and $v \in \mathcal{H}$ such that $x = u + Q^\# v$. Hence,

$$x = Qx = Q(u + Q^\# v) = QQ^\# v,$$

i.e. $x \in R(Q) \cap R(Q^\#)$. Thus, $R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$.

Conversely, suppose that $R(Q) \cap (N(Q)^\circ)^{[\perp]} \subseteq R(Q) \cap R(Q^\#)$. Then, consider the decomposition $Q = E + P$ given by Proposition 5.4, where $E, P \in L(\mathcal{H})$ are projections and $PP^\# = P^\# P = 0$. Observe that

$$R(E) = R(Q) \cap (N(Q)^\circ)^{[\perp]} = R(Q) \cap R(Q^\#),$$

because $N(Q)^\circ \subseteq N(Q) = R(Q^\#)^{[\perp]}$. Also,

$$R(E^\#) = N(E)^{[\perp]} = N(Q)^{[\perp]} \cap (R(Q)^\circ)^{[\perp]} \supseteq R(Q^\#) \cap R(Q) = R(E).$$

Thus, $E^\# E = E$ and, by Theorem 3.1, $Q$ is $J$-normal.

Finally, notice that the equivalence 1. $\leftrightarrow$ 3. follows considering $I - Q$ instead of $Q$. $\square$

The following result shows that, for a fixed $J$-neutral subspace, there are infinitely many $J$-normal projections onto it. Furthermore, the nullspaces of these projections can be arbitrarily close.

**Proposition 5.6** (T. Ando)**.** *Suppose that a (non-trivial) projection $P \in L(\mathcal{H})$ satisfies $PP^\# = P^\# P = 0$. Then, there exists a one-parameter family of (different) $J$-normal projections $P_\varepsilon \in L(\mathcal{H})$ onto $R(P)$ (for $0 < \varepsilon < \varepsilon_0$) such that*

$$\|P_\varepsilon - P\| \to 0 \quad \text{as } \varepsilon \to 0.$$

*Proof.* Let $P_R$ (resp. $P_N$) be the orthogonal projection onto $R(P)$ (resp. $N(P)$). Then, the ranges of these projections are $J$-neutral subspaces and, by Lemma 2.3, there is a partial isometry $V \in L(\mathcal{H}_+, \mathcal{H}_-)$ such that

$$I - P_N = \frac{1}{2}\begin{pmatrix} V^*V & V^* \\ V & VV^* \end{pmatrix}.$$

Since $e^{i\varepsilon}V$ is also a partial isometry (for every $\varepsilon > 0$), there is an orthogonal projection $Q_\varepsilon$ such that

$$I - Q_\varepsilon = \frac{1}{2}\begin{pmatrix} V^*V & e^{-i\varepsilon}V^* \\ e^{i\varepsilon}V & VV^* \end{pmatrix},$$

so that $(I - Q_\varepsilon)J(I - Q_\varepsilon) = 0$. It is clear that $\|P_N - Q_\varepsilon\| \to 0$ as $\varepsilon \to 0$.

Since $\|P_R P_N\| < 1$ and $\|(I-P_R)(I-P_N)\| < 1$, there exists $\varepsilon_0 > 0$ such that

$$\|P_R Q_\varepsilon\| < 1 \quad \text{and} \quad \|(I-P_R)(I-Q_\varepsilon)\| < 1 \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

Hence, there is a projection $P_\varepsilon \in L(\mathcal{H})$ with $R(P_\varepsilon) = R(P)$ and $N(P_\varepsilon) = R(Q_\varepsilon)$, see Remark 2.1. Then, by Lemma 2.3, $P_\varepsilon P_\varepsilon^\# = P_\varepsilon^\# P_\varepsilon = 0$. Finally, $P_\varepsilon$ can be represented as:

$$P_\varepsilon = P_R(P_R + Q_\varepsilon)^{-1},$$

see (2.1). So, $P_\varepsilon \neq P$ for every $0 < \varepsilon \leq \varepsilon_0$, and $\|P_\varepsilon - P\| \to 0$ as $\varepsilon \to 0$. $\square$

**Corollary 5.7.** *Suppose that a (non-trivial) projection $P \in L(\mathcal{H})$, satisfies $PP^\# = P^\# P = 0$. Then, there exists a one-parameter family of (different) $J$-normal projections $P_\varepsilon \in L(\mathcal{H})$ onto $R(P)$ (for $0 < \varepsilon < \varepsilon_0$) such that*

$$c(N(P), N(P_\varepsilon)) \longrightarrow 1 \quad \textit{as } \varepsilon \to 0.$$

*Proof.* Consider the projections $P_\varepsilon$ obtained in Proposition 5.6. Following the notations in the proof above, $N(P) = R(P_N)$ and $N(P_\varepsilon) = R(Q_\varepsilon)$. Then,

$$c(N(P), N(P_\varepsilon)) = c(R(P_N), R(Q_\varepsilon)) = c(R(I-P_N), R(I-Q_\varepsilon)),$$

because $P_N$ and $Q_\varepsilon$ are orthogonal projections. By Remark 2.1,

$$\begin{aligned}
& c(R(I-P_N), R(I-Q_\varepsilon))^2 = \\
= \;& \|(I-Q_\varepsilon)(I-P_N)\|^2 = \|(I-Q_\varepsilon)(I-P_N)(I-Q_\varepsilon)\| = \\
= \;& \frac{|(1+e^{i\varepsilon})(1+e^{-i\varepsilon})|}{4} \left\| \frac{1}{2} \begin{pmatrix} V^*V & \frac{1+e^{-i\varepsilon}}{1+e^{i\varepsilon}} V^* \\ \frac{1+e^{i\varepsilon}}{1+e^{-i\varepsilon}} V & VV^* \end{pmatrix} \right\| = \\
= \;& \frac{|(1+e^{i\varepsilon})(1+e^{-i\varepsilon})|}{4} = \frac{1+cos(\varepsilon)}{2} = \cos^2(\tfrac{\varepsilon}{2}).
\end{aligned}$$

Therefore, $c(N(P), N(P_\varepsilon)) = \cos(\frac{\varepsilon}{2}) \longrightarrow 1$ as $\varepsilon \to 0$. $\square$

## $J$-normal projections with prescribed $J$-neutral range

Let $\mathcal{N}$ be a $J$-neutral subspace of a Krein space $\mathcal{H}$ with fundamental symmetry $J$. Along these paragraphs, a parametrization for the set of $J$-normal projections onto $\mathcal{N}$ is presented. These results are generalized to an arbitrary pseudo-regular subspace in Section 6.

According to the orthogonal decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$, the symmetry $J$ can be written as a block-operator-matrix

$$J = \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} \begin{array}{l} \mathcal{N} \\ \mathcal{N}^\perp \end{array} \tag{5.1}$$

where $a \in L(\mathcal{N}^\perp, \mathcal{N})$ and $b = b^* \in L(\mathcal{N}^\perp)$ satisfy

$$aa^* = I_{\mathcal{N}}, \quad ab = 0 \quad \text{and} \quad a^*a + b^2 = I_{\mathcal{N}^\perp}. \tag{5.2}$$

Since $a \in L(\mathcal{N}^\perp, \mathcal{N})$ is a coisometry, it follows that $a^* \in L(\mathcal{N}, \mathcal{N}^\perp)$ is a partial isometry with final space:

$$R(a^*a) = R(a^*) = J(\mathcal{N}).$$

Thus, $a^*a \in L(\mathcal{N}^\perp)$ is the orthogonal projection onto $J(\mathcal{N})$.

On the other hand, if $P$ is a projection with range $\mathcal{N}$ then $P$ can be written as a block-operator-matrix

$$P = \begin{pmatrix} I & x \\ 0 & 0 \end{pmatrix},$$

with $x \in L(\mathcal{N}^\perp, \mathcal{N})$. Furthermore, $P$ satisfies $PP^\# = 0$ if and only if

$$0 = \begin{pmatrix} I & x \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ a^* & b \end{pmatrix} \begin{pmatrix} I & 0 \\ x^* & 0 \end{pmatrix} = \begin{pmatrix} ax^* + xa^* + xbx^* & 0 \\ 0 & 0 \end{pmatrix},$$

or equivalently, $x \in L(\mathcal{N}^\perp, \mathcal{N})$ is a solution of the equation

$$ax^* + xa^* + xbx^* = 0. \tag{5.3}$$

Thus, in order to describe the set of $J$-normal projections onto the $J$-neutral subspace $\mathcal{N}$, the above equation has to be solved. The following result provides a parametrization for the set of solutions of (5.3).

**Lemma 5.8.** *Let $\mathcal{N}$ be a $J$-neutral subspace of $\mathcal{H}$. Then, $x \in L(\mathcal{N}^\perp, \mathcal{N})$ is a solution of (5.3) if and only if there exist operators $A \in L(\mathcal{N})$ and $B \in L(\mathcal{N}^\perp, \mathcal{N})$ such that $A$ is antihermitian, $J(\mathcal{N}) \subseteq N(B)$ and*

$$x = (A - \tfrac{1}{2}BbB^*)a + B.$$

*Proof.* Recall that the operators $a$ and $b$ considered in (5.3) satisfy the conditions in (5.2). First, suppose that $x \in L(\mathcal{N}^\perp, \mathcal{N})$ is a solution of (5.3). Since $a^*a + b^2 = I_{\mathcal{N}^\perp}$, $x$ can be written as $x = x_1 + x_2$, where $x_1 = xa^*a$ and $x_2 = xb^2$.

Observe that $x_2a^* = x_1b = 0$. Thus, $0 = ax^* + xa^* + xbx^* = ax_1^* + x_1a^* + x_2bx_2^*$. In other words,

$$2\operatorname{Re}(x_1a^*) = ax_1^* + x_1a^* = -x_2bx_2^*.$$

So, the antihermitian operator $A = i\operatorname{Im}(x_1a^*) \in L(\mathcal{N})$ satisfies

$$x_1 = x_1a^*a = (A - \tfrac{1}{2}x_2bx_2^*)a.$$

Then, considering $B = x_2 = x(I_{\mathcal{N}^\perp} - a^*a) \in L(\mathcal{N}^\perp, \mathcal{N})$ it follows that $J(\mathcal{N}) \subseteq N(B)$ and

$$x = (A - \tfrac{1}{2}BbB^*)a + B.$$

Conversely, given an antihermitian operator $A \in L(\mathcal{N})$ and $B \in L(\mathcal{N}^\perp, \mathcal{N})$ such that $J(\mathcal{N}) \subseteq N(B)$, consider

$$x := (A - \tfrac{1}{2}BbB^*)a + B.$$

Then, it is easy to see that $xa^* = A - \frac{1}{2}BbB^*$ and $xbx^* = BbB^*$. Therefore,

$$xa^* + ax^* + xbx^* = (A - \tfrac{1}{2}BbB^*) + (-A - \tfrac{1}{2}BbB^*) + BbB^* = 0,$$

i.e. $x \in L(\mathcal{N}^\perp, \mathcal{N})$ is a solution of (5.3). $\square$

**Proposition 5.9.** *Let $\mathcal{N}$ be a $J$-neutral subspace of $\mathcal{H}$. Then, $P \in L(\mathcal{H})$ is a $J$-normal projection onto $\mathcal{N}$ if and only if there exist $A = -A^* \in L(\mathcal{N})$ and $B \in L(\mathcal{N}^\perp, \mathcal{N})$ with $J(\mathcal{N}) \subseteq N(B)$ such that*

$$P = \begin{pmatrix} I & (A - \frac{1}{2}BbB^*)a + B \\ 0 & 0 \end{pmatrix},$$

*according to the orthogonal decomposition $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}^\perp$.*

# 6 A parametrization for the set of $J$-normal projections

Let $\mathcal{S}$ be a pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundamental symmetry $J$, and denote

$$\mathcal{Q}_{\mathcal{S}} = \{Q \in L(\mathcal{H}) : \; Q^2 = Q, \; QQ^\# = Q^\# Q \text{ and } R(Q) = \mathcal{S}\}.$$

The aim of this section is to present an explicit parametrization of $\mathcal{Q}_{\mathcal{S}}$. First, notice that there are as many projections in $\mathcal{Q}_{\mathcal{S}}$ as in $\mathcal{Q}_{\mathcal{S}^\circ}$.

**Lemma 6.1.** *Suppose that $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$. If $P$ is a $J$-normal projection onto $\mathcal{S}^\circ$ then there is a unique $J$-normal projection $Q$ onto $\mathcal{S}$ such that $P$ is the neutral part of $Q$, i.e. $P = Q(I - Q)^\#$.*

*Proof.* Suppose that $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$ and consider $\mathcal{T} = \mathcal{S} \cap N(P)$. Since $P$ is a projection onto $\mathcal{S}^\circ \subseteq \mathcal{S}$, given $s \in \mathcal{S}$, $(I-P)s \in \mathcal{S} + \mathcal{S}^\circ = \mathcal{S}$. So that, $(I-P)s \in \mathcal{S} \cap N(P)$. Therefore,

$$\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{T}.$$

Then, by Proposition 4.1, $\mathcal{T}$ is a regular subspace of $\mathcal{H}$. Let $E$ be the $J$-selfadjoint projection onto $\mathcal{T}$.

Notice that $EP = 0$ because $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]} \subseteq \mathcal{T}^{[\perp]}$. On the other hand, $R(E) = \mathcal{T} \subseteq N(P)$. So, $PE = 0$ and, since $E$ is $J$-selfadjoint, the following commutativity relations have been established:

$$EP = PE = 0 \quad \text{and} \quad EP^\# = P^\# E = 0.$$

Now, define $Q = E + P$. Then, by Theorem 3.1, $Q$ is a $J$-normal projection and $P = Q - E = Q - QQ^\# = Q(I - Q^\#)$.

Finally, suppose that there is another $J$-normal projection $Q' \in L(\mathcal{H})$ onto $\mathcal{S}$ such that $P = Q'(I - Q')^\#$. Then, $E' = Q' - P = Q'(Q')^\#$ is a $J$-selfadjoint projection onto a subspace of $\mathcal{S}$. Notice that $R(E') \subseteq N(P)$ because $PE' = 0$. Hence, $R(E') \subseteq \mathcal{T}$. But,

$$R(E') \dotplus \mathcal{S}^\circ = \mathcal{S} = \mathcal{T} \dotplus \mathcal{S}^\circ.$$

Thus, $R(E') = \mathcal{T}$ and, by the uniqueness of the $J$-selfadjoint projection onto a regular subspace, $E' = E$. □

**Theorem 6.2.** *Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$ with isotropic part $\mathcal{S}^\circ$, there is a (continuous) bijection between $\mathcal{Q}_\mathcal{S}$ and $\mathcal{Q}_{\mathcal{S}^\circ}$.*

*Proof.* For a fixed pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, let $\Phi : \mathcal{Q}_\mathcal{S} \to \mathcal{Q}_{\mathcal{S}^\circ}$ be defined by

$$\Phi(Q) = Q(I - Q^\#).$$

It follows by the above lemma that $\Phi$ is bijective, because for every $P \in \mathcal{Q}_{\mathcal{S}^\circ}$ there exists a unique $Q \in \mathcal{Q}_\mathcal{S}$ such that $\Phi(Q) = P$. □

**Corollary 6.3.** *Let $\mathcal{S}$ be a pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundamental symmetry $J$. Then, there is a unique $J$-normal projection $Q$ onto $\mathcal{S}$ if and only if $\mathcal{S}^\circ = \{0\}$. Moreover, in this case $Q$ is $J$-selfadjoint.*

*Proof.* If $\mathcal{S}^\circ = \{0\}$ then $\mathcal{S}$ is a regular subspace and there exists a (unique) $J$-selfadjoint projection onto $\mathcal{S}$. Moreover, if $Q$ is a $J$-normal projection onto $\mathcal{S}$ then, by Theorem 3.1, $Q = E + P$ where $E$ is $J$-selfadjoint and $P$ is a projection onto $\mathcal{S}^\circ = \{0\}$. Thus, $P = 0$ and $Q = E$.

On the other hand, if $\mathcal{S}^\circ \neq \{0\}$ then, as a consequence of Theorem 6.2 and Proposition 5.6, there are infinitely many $J$-normal projections onto $\mathcal{S}$. □

By Proposition 4.1, for a fixed pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, if $\mathcal{S}^\circ$ is the isotropic part of $\mathcal{S}$ and $\mathcal{M}$ is a subspace of $\mathcal{S}$ such that $\mathcal{S} = \mathcal{S}^\circ [\dotplus] \mathcal{M}$ (i.e. $\mathcal{M}$ is a complement of $\mathcal{S}^\circ$ in $\mathcal{S}$), then $\mathcal{M}$ is a regular subspace of $\mathcal{H}$. Hence, consider

$$\mathcal{Q}_{\mathcal{S},\mathcal{M}} = \{Q \in \mathcal{Q}_\mathcal{S} : \ QQ^\# = E_\mathcal{M}\},$$

where $E_\mathcal{M}$ stands for the $J$-selfadjoint projection onto $\mathcal{M}$.

Notice that $\mathcal{Q}_\mathcal{S}$ can be written as the disjoint union of the family $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$, as $\mathcal{M}$ varies on the complements of $\mathcal{S}^\circ$ in $\mathcal{S}$:

**Lemma 6.4.** *If $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$, then*

$$\mathcal{Q}_\mathcal{S} = \dot{\bigcup}_{\{\mathcal{M}:\ \mathcal{S} = \mathcal{S}^\circ [\dotplus] \mathcal{M}\}} \mathcal{Q}_{\mathcal{S},\mathcal{M}}, \tag{6.1}$$

*where $\dot\cup$ denotes a disjoint union.*

*Proof.* It is obvious that $\mathcal{Q}_\mathcal{S} = \bigcup_{\{\mathcal{M}:\ \mathcal{S} = \mathcal{S}^\circ [\dotplus] \mathcal{M}\}} \mathcal{Q}_{\mathcal{S},\mathcal{M}}$. Suppose that $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}_1} \cap \mathcal{Q}_{\mathcal{S},\mathcal{M}_2}$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are regular subspaces of $\mathcal{H}$. Then,

$$E_{\mathcal{M}_1} = QQ^\# = E_{\mathcal{M}_2},$$

or equivalently, $\mathcal{M}_1 = \mathcal{M}_2$. Hence, $\mathcal{Q}_{\mathcal{S},\mathcal{M}_1} = \mathcal{Q}_{\mathcal{S},\mathcal{M}_2}$. □

## Parametrizing the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ for a pseudo-regular subspace $\mathcal{S}$

The following paragraphs are devoted to studying those $J$-normal projections onto $\mathcal{S}$ which have a fixed regular part. Along this section operators are treated as block-operator matrices according to the orthogonal decomposition

$$\mathcal{H} = \mathcal{S}^\circ \oplus (\mathcal{S} \ominus \mathcal{S}^\circ) \oplus \mathcal{S}^\perp,$$

and $P_{\mathcal{S}^\perp}$, $P_{\mathcal{S}^\circ}$ and $P_{\mathcal{S}\ominus\mathcal{S}^\circ}$ denote the orthogonal projections onto $\mathcal{S}^\perp$, $\mathcal{S}^\circ$ and $\mathcal{S} \ominus \mathcal{S}^\circ$, respectively.

If $\mathcal{M}$ is a regular subspace of $\mathcal{H}$ such that $\mathcal{S} = \mathcal{S}^\circ[\dot{+}]\mathcal{M}$, it is necessary to describe the fundamental symmetry $J$ and the $J$-selfadjoint projection $E_\mathcal{M}$ onto $\mathcal{M}$ as block-operator matrices.

**Lemma 6.5.** *If $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$, then $J$ is represented as the block-operator matrix*

$$J = \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix} \begin{matrix} \mathcal{S}^\circ \\ \mathcal{S} \ominus \mathcal{S}^\circ \\ \mathcal{S}^\perp \end{matrix} \ , \tag{6.2}$$

*where $a \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$, $b = b^* \in GL(\mathcal{S} \ominus \mathcal{S}^\circ)$, $c \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$ and $d = d^* \in L(\mathcal{S}^\perp)$ satisfy the following equations:*

$$\left\{ \begin{array}{rcl} aa^* & = & I_{\mathcal{S}^\circ} \\ b^2 + cc^* & = & I_{\mathcal{S}\ominus\mathcal{S}^\circ} \\ a^*a + c^*c + d^2 & = & I_{\mathcal{S}^\perp} \\ bc + cd = ad = ac^* & = & 0 \end{array} \right. . \tag{6.3}$$

*Proof.* Notice that $P_{\mathcal{S}^\circ} J P_{\mathcal{S}^\circ} = 0$ because $\mathcal{S}^\circ$ is $J$-neutral. Also, $P_{\mathcal{S}^\circ} J P_{\mathcal{S}\ominus\mathcal{S}^\circ} = 0$ because $\mathcal{S} \ominus \mathcal{S}^\circ \subseteq \mathcal{S}$ and $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]}$. Then,

$$J = \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix}.$$

On the other hand, the system of equations (6.3) follows from $J^2 = I$.

By Proposition 4.1, $\mathcal{S} \ominus \mathcal{S}^\circ$ is a regular subspace of $\mathcal{H}$. Furthermore, the regularity of $\mathcal{S} \ominus \mathcal{S}^\circ$ is equivalent to the range inclusion

$$R(c) \subseteq R(b),$$

see [7, Prop. 3.3]. Then, the second equation in (6.3) implies that $\mathcal{S}\ominus\mathcal{S}^\circ \subseteq R(b)$. Hence, $b$ is an invertible selfadjoint operator in $L(\mathcal{S} \ominus \mathcal{S}^\circ)$. $\square$

*Remark* 6.6. Observe that the operator $a \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ appearing in the above lemma is a coisometry. Then, $a^* \in L(\mathcal{S}^\circ, \mathcal{S}^\perp)$ is a partial isometry with final space $J(\mathcal{S}^\circ)$.

Indeed, by the block-operator matrix representation of $J$ given in (6.2), it is easy to see that $R(a^*) = J(\mathcal{S}^\circ)$. Hence,

$$R(a^*a) = R(a^*) = J(\mathcal{S}^\circ). \tag{6.4}$$

Thus, $a^*a \in L(\mathcal{S}^\perp)$ is the orthogonal projection onto $J(\mathcal{S}^\circ)$.

The following lemma presents a block-matrix representation for the $J$-selfadjoint projection $E_\mathcal{M}$ onto a particular complement $\mathcal{M}$ of $\mathcal{S}^\circ$ in $\mathcal{S}$. This is a technical tool necessary to parametrize the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$.

**Lemma 6.7.** *Given a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$, let $\mathcal{M}$ be a complement of $\mathcal{S}^\circ$ in $\mathcal{S}$. Then, the $J$-selfadjoint projection onto $\mathcal{M}$ is*

$$E_\mathcal{M} = \begin{pmatrix} 0 & ar^*b & ar^*(c+br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.5}$$

*where $r = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_\mathcal{M} P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ)$.*

*Proof.* Suppose that $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$. Then, by Proposition 4.1, $\mathcal{M}$ is regular.

Denote by $E_\mathcal{M}$ the $J$-selfadjoint projection onto $\mathcal{M}$. Since $R(E_\mathcal{M}) = \mathcal{M} \subseteq \mathcal{S}$ it follows that $P_{\mathcal{S}^\perp} E_\mathcal{M} = 0$, so that the third row in the matrix representation of $E_\mathcal{M}$ is zero. Also, since $\mathcal{S}^\circ \subseteq \mathcal{S}^{[\perp]} \subseteq \mathcal{M}^{[\perp]} = N(E_\mathcal{M})$, it follows that $E_\mathcal{M} P_{\mathcal{S}^\circ} = 0$. So that the first column is also zero. Therefore,

$$E_\mathcal{M} = \begin{pmatrix} 0 & u & v \\ 0 & p & q \\ 0 & 0 & 0 \end{pmatrix},$$

where $u \in L(\mathcal{S}\ominus\mathcal{S}^\circ, \mathcal{S}^\circ)$, $v \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$, $p \in L(\mathcal{S}\ominus\mathcal{S}^\circ)$ and $q \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ)$ satisfy

$$\begin{cases} up &= u \\ uq &= v \\ p^2 &= p \\ pq &= q \end{cases}.$$

Thus, $p = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_\mathcal{M}|_{\mathcal{S}\ominus\mathcal{S}^\circ}$ is a projection with

$$R(p) = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_\mathcal{M}(\mathcal{S}\ominus\mathcal{S}^\circ) = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_\mathcal{M}(\mathcal{S}) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}(\mathcal{M}) = P_{\mathcal{S}\ominus\mathcal{S}^\circ}(\mathcal{S}) = \mathcal{S}\ominus\mathcal{S}^\circ,$$

because $\mathcal{S}^\circ \subseteq N(P_{\mathcal{S}\ominus\mathcal{S}^\circ}) \cap N(E_\mathcal{M})$. Hence, $p = I_{\mathcal{S}\ominus\mathcal{S}^\circ}$.

Furthermore, $E_\mathcal{M}$ is $J$-selfadjoint if and only if

$$JE_\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & bq \\ 0 & a^*u + c^* & (a^*u + c^*)q \end{pmatrix}$$

is selfadjoint, or equivalently, if

$$a^*u + c^* = q^*b. \tag{6.6}$$

By (6.3), $aa^* = I_{\mathcal{S}^\circ}$ and $ac^* = 0$. Thus, multiplying on the left by $a$, it follows that $u = aq^*b$. Thus,

$$E_{\mathcal{M}} = \begin{pmatrix} 0 & aq^*b & aq^*bq \\ 0 & I & q \\ 0 & 0 & 0 \end{pmatrix},$$

where $q = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_{\mathcal{M}}|_{\mathcal{S}^\perp}$. Replacing $u$ is (6.6), notice that $q$ satisfies $a^*aq^*b + c^* = q^*b$, or equivalently,

$$q = q(a^*a) + b^{-1}c.$$

Therefore, if $r = q(a^*a)$ then $aq^*b = a(c^*b^{-1}+r^*)b = ar^*b$, and (6.5) follows. $\square$

Finally, a block-matrix representation of a projection $Q \in L(\mathcal{H})$ onto $\mathcal{S}$ is needed. Since $R(Q) = \mathcal{S}$, observe that $P_{\mathcal{S}^\circ} Q P_{\mathcal{S}^\circ} = P_{\mathcal{S}^\circ}$, $P_{\mathcal{S}\ominus\mathcal{S}^\circ} Q P_{\mathcal{S}\ominus\mathcal{S}^\circ} = P_{\mathcal{S}\ominus\mathcal{S}^\circ}$ and

$$P_{\mathcal{S}^\circ} Q P_{\mathcal{S}\ominus\mathcal{S}^\circ} = P_{\mathcal{S}\ominus\mathcal{S}^\circ} Q P_{\mathcal{S}^\circ} = 0.$$

Then, $Q$ is represented as the block-operator matrix

$$Q = \begin{pmatrix} I & 0 & x \\ 0 & I & y \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.7}$$

where $x = P_{\mathcal{S}^\circ} Q|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ and $y = P_{\mathcal{S}\ominus\mathcal{S}^\circ} Q|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ)$.

Furthermore, if $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$ then, by Theorem 3.1, $P = Q - E_{\mathcal{M}}$ is a projection onto $\mathcal{S}^\circ$ such that $PP^\# = P^\# P = 0$. Moreover, by (6.5), $P$ has the form

$$P = Q - E_{\mathcal{M}} = \begin{pmatrix} I & -ar^*b & x - ar^*(c+br) \\ 0 & 0 & y - b^{-1}c - r \\ 0 & 0 & 0 \end{pmatrix}.$$

But, $R(P) = \mathcal{S}^\circ$ if and only if

$$y = b^{-1}c + r.$$

Also, $PP^\# = 0$ if and only if $PJP^* = 0$, or equivalently,

$$\begin{pmatrix} I & -ar^*b & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & a \\ 0 & b & c \\ a^* & c^* & d \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -bra^* & 0 & 0 \\ z^* & 0 & 0 \end{pmatrix} = 0,$$

where $z = x - ar^*(c+br)$. But the above equation is equivalent to

$$z(I - r^*bc)^*a^* + a(I - r^*bc)z^* + zdz^* + ar^*b^3ra^* = 0. \tag{6.8}$$

The following lemma is devoted to describe the solutions of (6.8), where $a$, $b$, $c$, $d$ and $r$ are the operators appearing in (6.2) and in (6.5).

**Lemma 6.8.** *An operator $z \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is a solution of (6.8) if and only if there exist $A = -A^* \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ with $J(\mathcal{S}^\circ) \subseteq N(B)$ such that*

$$z = (A + \operatorname{Re}(Bc^*bra^*) - \tfrac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

*Proof.* Let $z \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ be a solution of (6.8) and consider the operators

$$z_1 = z(a^*a) \quad \text{and} \quad z_2 = z(I_{\mathcal{S}^\perp} - a^*a).$$

Notice that $z_1(I - r^*bc)^*a^* + a(I - r^*bc)z_1^* = z_1a^* + az_1^* = 2\operatorname{Re}(z_1a^*)$ because $ac^* = ca^* = 0$. Also,

$$z_2(I - r^*bc)^*a^* + a(I - r^*bc)z_2^* = -z_2c^*bra^* - ar^*bcz_2^* = -2\operatorname{Re}(z_2c^*bra^*),$$

because $z_2a^* = az_2^* = 0$. On the other hand, since $ad = da^* = 0$ it is easy to see that

$$zdz^* = (z_1 + z_2)d(z_1 + z_2)^* = z_2dz_2^*.$$

Therefore, (6.8) is equivalent to

$$2\operatorname{Re}(z_1a^*) = 2\operatorname{Re}(z_2c^*bra^*) - z_2dz_2^* - ar^*b^3ra^*. \tag{6.9}$$

Then, considering the antihermitian operator $A = i\operatorname{Im}(z_1a^*) \in L(\mathcal{S}^\circ)$, it follows that

$$\begin{aligned} z_1 &= (z_1a^*)a = (i\operatorname{Im}(z_1a^*) + \operatorname{Re}(z_1a^*))a \\ &= (A + \operatorname{Re}(z_2c^*bra^*) - \tfrac{1}{2}(z_2dz_2^* + ar^*b^3ra^*))a. \end{aligned}$$

Hence, $B = z_2 \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ satisfies $J(\mathcal{S}^\circ) \subseteq N(B)$ and

$$z = z_1 + z_2 = (A + \operatorname{Re}(Bc^*bra^*) - \tfrac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

Conversely, given an antihermitian operator $A \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ such that $N(b)^\perp \subseteq N(d)$, consider

$$z_{A,B} := (A + \operatorname{Re}(Bc^*bra^*) - \tfrac{1}{2}(BdB^* + ar^*b^3ra^*))a + B.$$

Then, it is easy to see that $z_{A,B} \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is a solution of (6.8). $\square$

Finally, it is possible to parametrize the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ as follows:

**Theorem 6.9.** *Let $Q \in L(\mathcal{H})$ be a projection onto a pseudo-regular subspace $\mathcal{S}$ of $\mathcal{H}$. Suppose that $\mathcal{M}$ is a regular subspace of $\mathcal{H}$ such that $\mathcal{S} = \mathcal{S}^\circ \dot{+} \mathcal{M}$. Then, $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$ if and only if*

$$Q = \begin{pmatrix} I & 0 & (A + \operatorname{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c + br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.10}$$

*where $r = P_{\mathcal{S}\ominus\mathcal{S}^\circ}E_{\mathcal{M}}(a^*a) \in L(\mathcal{S}^\perp, \mathcal{S} \ominus \mathcal{S}^\circ)$, $A = -A^* \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is such that $J(\mathcal{S}^\circ) \subseteq N(B)$.*

*Proof.* Suppose that $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{M}}$, i.e. $Q \in L(\mathcal{H})$ is a $J$-normal projection onto $\mathcal{S}$ satisfying $QQ^{\#} = Q^{\#}Q = E_{\mathcal{M}}$. Then, $P = Q - E_{\mathcal{M}}$ is a projection onto $\mathcal{S}^{\circ}$ such that $PP^{\#} = P^{\#}P = 0$. Hence, if $Q$ is written as in (6.7) it follows that $y = b^{-1}c$.

Then, by the discussion above,

$$P = \begin{pmatrix} I & -ar^*b & x - ar^*(c+br) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $z = x - ar^*(c+br)$ is a solution of (6.8). Thus, by Proposition 6.8, there exist $A = -A^* \in L(\mathcal{S}^{\circ})$ and $B \in L(\mathcal{S}^{\perp}, \mathcal{S}^{\circ})$ with $J(\mathcal{S}^{\circ}) \subseteq N(B)$ such that

$$P = \begin{pmatrix} I & -ar^*b & (A + \mathrm{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$Q = \begin{pmatrix} I & 0 & (A + \mathrm{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c+br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix}.$$

The converse follows immediately. $\square$

Given a pseudo regular subspace $\mathcal{S}$ of $\mathcal{H}$, denote by $\mathcal{C}(\mathcal{S}^{\circ})$ the set of complements of $\mathcal{S}^{\circ}$ in $\mathcal{S}$. Recall that, by Lemma 6.4, the set of $J$-normal projections onto $\mathcal{S}$ is decomposed as

$$\mathcal{Q}_{\mathcal{S}} = \dot{\bigcup}_{\mathcal{M} \in \mathcal{C}(\mathcal{S}^{\circ})} \mathcal{Q}_{\mathcal{S},\mathcal{M}}.$$

Furthermore, for a fixed $\mathcal{M} \in \mathcal{C}(\mathcal{S}^{\circ})$, Theorem 6.9 states that the deck $\mathcal{Q}_{\mathcal{S},\mathcal{M}}$ is parametrized by the bijection $\Psi_{\mathcal{M}} : \mathcal{AH}(\mathcal{S}^{\circ}) \times \mathcal{N}_{\circ} \to \mathcal{Q}_{\mathcal{S},\mathcal{M}}$ given by

$$\Psi_{\mathcal{M}}(A,B) = \begin{pmatrix} I & 0 & (A + \mathrm{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c+br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix},$$

where $\mathcal{AH}(\mathcal{S}^{\circ})$ stands for the real vector space of antihermitian operators acting on $\mathcal{S}^{\circ}$ and $\mathcal{N}_{\circ}$ is the set composed by those operators $B \in L(\mathcal{S}^{\perp}, \mathcal{S}^{\circ})$ such that $J(\mathcal{S}^{\circ}) \subseteq N(B)$.

Therefore, the set $\mathcal{Q}_{\mathcal{S}}$ of $J$-normal projections onto $\mathcal{S}$ is parametrized as follows:

**Theorem 6.10.** *Let $\mathcal{S}$ be a pseudo-regular subspace of $\mathcal{H}$. Then, the function $\Psi : \mathcal{RC}(\mathcal{S}^{\circ}) \times \mathcal{AH}(\mathcal{S}^{\circ}) \times \mathcal{N}_{\circ} \to \mathcal{Q}_{\mathcal{S}}$ defined by*

$$\Psi(\mathcal{M}, A, B) = \begin{pmatrix} I & 0 & (A + \mathrm{Re}(Bc^*bra^*) - \frac{1}{2}(BdB^* + ar^*b^3ra^*))a + B + ar^*(c+br) \\ 0 & I & b^{-1}c + r \\ 0 & 0 & 0 \end{pmatrix},$$

*is one-to one.*

Observe that in the expression defining $\Psi$ appears the operator

$$r = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E_{\mathcal{M}} P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ),$$

given in Lemma 6.7, where $P_{\mathcal{S}\ominus\mathcal{S}^\circ}$ and $P_{J(\mathcal{S}^\circ)}$ are the orthogonal projections onto $\mathcal{S}\ominus\mathcal{S}^\circ$ and $J(\mathcal{S}^\circ)$, respectively, and $E_{\mathcal{M}}$ is the $J$-selfadjoint projection onto $\mathcal{M}$.

## An interesting particular deck: $\mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$

Let $\mathcal{S}$ be a fixed pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundametal symmetry $J$. These paragraphs are devoted to describe the set $\mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$, i.e. the family of $J$-normal projections $Q \in L(\mathcal{H})$ onto $\mathcal{S}$ such that $QQ^\#$ is the $J$-selfadjoint projection onto the (regular) subspace $\mathcal{S}\ominus\mathcal{S}^\circ$. In this particular deck there is a minimal norm projection, see Remark 6.12.

First of all, since $\mathcal{S}\ominus\mathcal{S}^\circ$ is a complement of $\mathcal{S}^\circ$ in $\mathcal{S}$, it follows by Lemma 6.7 that the $J$-selfadjoint projection onto $\mathcal{S}\ominus\mathcal{S}^\circ$ (hereafter denoted by $E$) is the block-operator matrix given by (6.5), where

$$r = P_{\mathcal{S}\ominus\mathcal{S}^\circ} E P_{J(\mathcal{S}^\circ)}|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S}\ominus\mathcal{S}^\circ).$$

But, $J(\mathcal{S}^\circ) \subseteq J(\mathcal{S}^\circ) + \mathcal{S}^{[\perp]} = J(\mathcal{S}^\circ + \mathcal{S}^\perp) = J((\mathcal{S}\ominus\mathcal{S}^\circ)^\perp) = N(E)$. Therefore, $r = 0$ and the block-operator matrix representation of $E$ is

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & b^{-1}c \\ 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, as a consequence of Theorem 6.9, $\mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$ is parametrized as:

**Proposition 6.11.** *Let $\mathcal{S}$ be a pseudo-regular subspace of a Krein space $\mathcal{H}$ with fundametal symmetry $J$. A projection $Q$ onto $\mathcal{S}$ satisfies $QQ^\# = Q^\# Q = E$ if and only if*

$$Q = \begin{pmatrix} I & 0 & (A - \frac{1}{2}BdB^*)a + B \\ 0 & I & b^{-1}c \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.11}$$

*where $a$, $b$, $c$ and $d$ are the operators appearing in* (6.2), *$A = -A^* \in L(\mathcal{S}^\circ)$ and $B \in L(\mathcal{S}^\perp, \mathcal{S}^\circ)$ is such that $J(\mathcal{S}^\circ) \subseteq N(B)$.*

*Remark* 6.12. In this particular case it is possible to estimate

$$\min\{\|Q\| : Q \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}\}.$$

Indeed, if $P_0$ is the orthogonal projection onto $\mathcal{S}^\circ$ and $E$ stands for the $J$-selfadjoint projection onto $\mathcal{S}\ominus\mathcal{S}^\circ$, then $Q_0 = E + P_0 \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$. Furthermore,

$$\|Q_0\|^2 = \|Q_0 Q_0^*\| = \|EE^* + P_0\| = \max\{\|EE^*\|, \|P_0\|\} = \|EE^*\| = \|E\|^2,$$

because $R(EE^*) = \mathcal{S} \ominus \mathcal{S}^\circ$ is orthogonal to $R(P_0) = \mathcal{S}^\circ$. Therefore, $\|Q_0\| = \|E\|$.

On the other hand, if $Q \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}$ then there exists a (unique) $P = P^2 \in L(\mathcal{H})$ such that $PP^\# = P^\# P = 0$ and $Q = E + P$.

Consider a sequence $\{x_n\}_{n\geq 1}$ in the unit ball of $\mathcal{H}$ such that $\|Ex_n\| \to \|E\|$ as $n \to \infty$. Then,

$$\|Q\|^2 \geq \|Qx_n\|^2 = \|Ex_n\|^2 + \|Px_n\|^2 \geq \|Ex_n\|^2 \to \|E\|^2 = \|Q_0\|^2.$$

Hence, $\|Q_0\| = \min\{\|Q\| : \ Q \in \mathcal{Q}_{\mathcal{S},\mathcal{S}\ominus\mathcal{S}^\circ}\}$.


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